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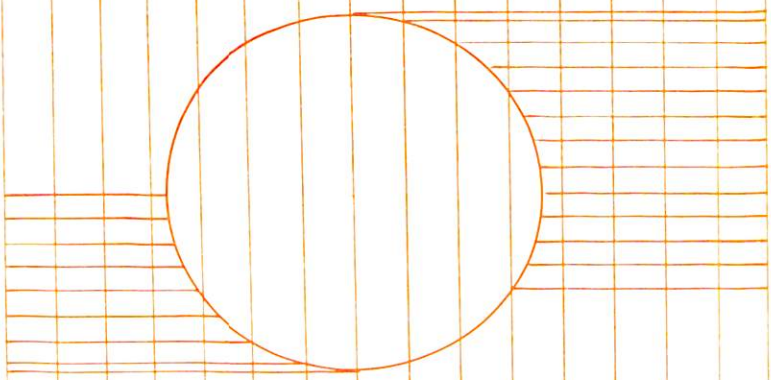

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#3

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Torsion

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1975

CHATTO & WINDUS
LONDON

Published by
Chatto & Windus Ltd
42 William IV Street
London WC2N 4DF

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Clarke, Irwin & Co. Ltd
Toronto

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ISBN 0 7011 1952 7

© A. C. Walker 1975

Printed in Great Britain by
T. & A. Constable Ltd
Hopetoun Street
Edinburgh EH7 4NF

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1

INTRODUCTION

This book is concerned with the manner in which a variety of structural elements carry torsional loading. According to the mathematical theory of deformable bodies all of the cross-sections discussed here could be treated exactly using the same analysis. But this is a complicated procedure and in this introductory text we have taken the approach of treating the thin-walled and thick-walled sections separately. By making appropriate assumptions the analyses are then simplified but nevertheless retaining an accuracy adequate for most engineering purposes. Also, wherever possible, indication has been given of the limiting loads to which the different types of cross-section shapes can be subjected and their mode of failure is discussed.

By limiting attention to structural elements which have no variation of torque along their length, and by making appropriate assumptions regarding the deformations, the torsional loads induce a system of two-dimensional plane shear stress in the material. That is, on an element of material, shown in Fig. 1.1 (a), we will have only shear stresses τ_1 and τ_2 active. If τ_1 acts to resist the torque then it must be accompanied by the *complementary shear** τ_2 and from the equilibrium of the element we have $\tau_2 = \tau_1$. The shear stresses will cause the deformation shown in Fig. 1.1 (b) and the relationship between stress τ and strain γ is

$$\tau = G\gamma, \tag{1.1}$$

where G is a material constant called the *Shear Modulus*.

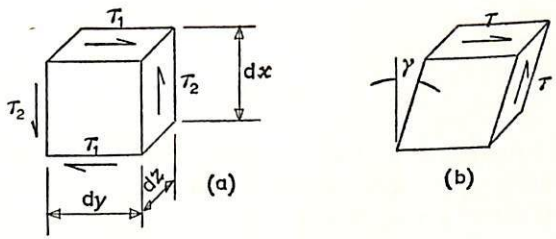


Fig. 1.1

* See T. R. Graves Smith, *Stress and Strain* (Chatto and Windus).

Consider any small change $d\gamma$ of the deformation of the element in Fig. 1.1, the work done by the shear forces is

$$dw = \tau \, d\gamma \, dx \, dy \, dz.$$

This must be stored as strain energy so that the change of strain energy is

$$du = \tau \, d\gamma \, dx \, dy \, dz,$$

and with (1.1) we can obtain the change in strain energy as the element is loaded,

$$dU = \int_0^{\gamma} \tau \, d\gamma \, dx \, dy \, dz = \frac{\tau^2}{2G} dx \, dy \, dz. \tag{1.2}$$

$\tau = G\gamma$

$$dU = \int_0^{\tau/G} \tau \, d(\tau/G) \, dx \, dy \, dz$$

$$= \frac{1}{G} \frac{\tau^2}{2} dx \, dy \, dz \Big|_0^{\tau/G}$$

2

TORSION OF THIN-WALLED TUBES

For a variety of reasons, such as low weight or material saving, many elements of structures are thin-walled. Often they are formed from sheet metal which is rolled and then seam welded; but there are examples such as the fuselage or wing of an aircraft, or the hull of a long oil tanker, that are much more complicated in construction but may also be considered, at least for initial design analysis, as thin-walled tubes. In this chapter we develop a theory to describe the manner in which such structures carry torsion loading. It is not an exact theory, but within the limitations of the assumptions stated later it is adequate for most engineering purposes. We start by examining the particular case of a circular cross-section and then go on to develop the ideas introduced there to apply to more general cross-sectional shapes, such as square or rectangular tubes.

2.1 Circular thin-walled tube

Consider the circular tube shown in Fig. 2.1; it has constant thickness t; R is the radius of the mid-thickness and the cross-section is uniform along the length L. The thickness is much less than the radius, and the tube is subjected to a torque T and has no longitudinal restraint.

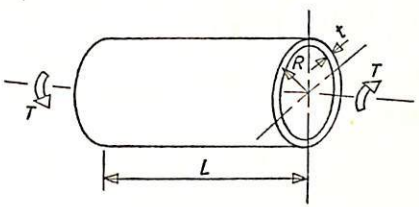


Fig. 2.1

We assume that the wall is sufficiently thin for us to ignore any variation of stresses across the thickness, and we can then say that the torque is the resultant of shear stresses τ, shown in Fig. 2.2 (a), that act tangentially to the centre line. If we now isolate an element of the tube, as shown in Fig. 2.2 (b), it is apparent that complementary stresses will be set up along the length of the tube. Equilibrium requires that the resultant of these

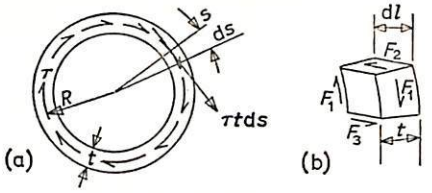


Fig. 2.2

complementary stresses be zero that is $F_2 = F_3$, but $F_2 = \tau t$ and $F_3 = (\tau + d\tau)t$, hence $d\tau = 0$ which indicates that the shear stress is everywhere uniform. The resultant force on the elementary portion ds of the periphery is $\tau t\, ds$, tangential to the centre line, hence the torque resultant is

$$T = \int_0^s \tau t R \, ds,$$

and since τ is uniform round the tube

$$T = 2\pi R^2 t\tau. \tag{2.1}$$

Thus

$$\tau = \frac{T}{2At}, \tag{2.2}$$

where A is the area enclosed by the centre line of the tube.

Having established the relationship between the stresses and torque resultant, we can now consider the deformation consequent on the application of the torque.

The shear and complementary shear stresses will cause an element of the tube to deform as indicated in Fig. 2.3 (a), and therefore the faces of an elementary length of the tube, Fig. 2.3 (b), will rotate relatively by $d\theta$. The angle γ is related to the stress τ by (1.1) hence,

$$R\, d\theta = \gamma\, dl = \frac{\tau}{G} dl,$$

and the rate of twist is

$$\frac{d\theta}{dl} = \frac{\tau}{GR}. \tag{2.3}$$

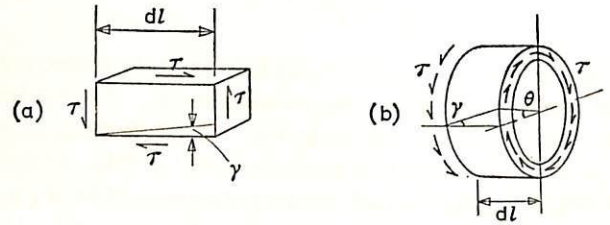


Fig. 2.3

For this particular tube the geometry and torque, and hence the stresses and rate of twist, are constant along the length. We can therefore integrate (2.3) to obtain the twist of one end of the tube relative to the other,

$$\int_0^\theta \mathrm{d}\theta = \int_0^l \frac{\tau}{GR}\,\mathrm{d}l,$$

that is

$$\theta = \frac{\tau L}{GR} = \frac{TL}{2AtGR}. \tag{2.4}$$

An alternative approach that is very useful for more complicated geometries, and which we apply here merely by way of introduction, is to use energy (see Ref. 3). The energy of deformation of the element in Fig. 2.3 (b) is

$$\mathrm{d}U = \tfrac{1}{2}t\tau\gamma\,\mathrm{d}l\,\mathrm{d}s = \frac{t}{2}\frac{\tau^2}{G}\,\mathrm{d}l\,\mathrm{d}s.$$

Thus, the energy required to deform the whole tube is

$$U = \int_0^L \int_0^s \frac{t\tau^2}{2G}\,\mathrm{d}l\,\mathrm{d}s = \frac{\pi R\tau^2 Lt}{G}.$$

The corresponding work done by the applied torque is $\frac{1}{2}T\theta$; hence equating the work done and the elastic energy, and using (2.2),

$$\theta = \frac{TL}{2AGRt}, \tag{2.5}$$

which is of course identical to (2.4). Notice we have made the very important implicit assumption that *the shape of the cross-section is unaltered during loading*. Moreover, in this particular example of a circular tube we will have plane cross-sections remaining plane. This follows from the symmetry of the section and is not generally true for other shapes.

2.2 Non-circular thin-walled tube

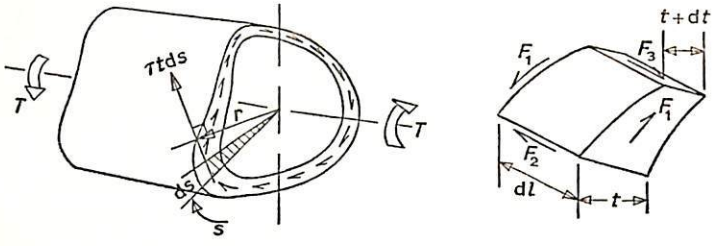
Now we will consider the more general problem of a non-circular tube subject to torsion loading. We also allow the thickness to vary but maintain the requirement that everywhere it must be small relative to the overall cross-section dimensions. We assume that **the wall is sufficiently thin that there is no variation of shear stress across the thickness**, also no longitudinal restraint exists so that the only stresses on the cross-section are shear stresses acting in a direction tangential to a line running around the mid-thickness of the perimeter of the section. Consider an elementary length of this mid-thickness line shown in Fig. 2.4 (a); the resultant force of the

shear stress is $\tau t\,\mathrm{d}s$ and the moment about the axis is $r\tau t\,\mathrm{d}s$. Hence, the resultant torque is

$$T = \int_0^s r\tau t\,\mathrm{d}s. \tag{2.6}$$

The shear stresses will vary around the centre line, but equilibrium requirements for the small element of tube, shown in Fig. 2.4 (b) indicate that $F_2 = F_3$. But, $F_2 = \tau t$, $F_3 = (\tau+\mathrm{d}\tau)(t+\mathrm{d}t)$, so that if we neglect the secondary term $\mathrm{d}\tau\,\mathrm{d}t$, then $\mathrm{d}(t\tau) = 0$ and the product $t\tau$ must be constant.* Hence (2.6) can be written

$$T = \tau t \int_0^s r\,\mathrm{d}s.$$



(a) (b)

Fig. 2.4

If we measure the periphery from any arbitrary position we see from Fig. 2.4 (a) that $r\,\mathrm{d}s$ will equal twice the shaded area and so $\int_0^s r\,\mathrm{d}s$ is twice the area enclosed by the centre line of the tube cross-section. That is,

$$T = 2A\tau t.$$

Applying the energy method to determine the deformation of the tube, and assuming once more that no distortion of the cross-section takes place,

$$\mathrm{d}U = \frac{t\tau^2}{2G}\,\mathrm{d}s\,\mathrm{d}l,$$

hence

$$U = \frac{(t\tau)^2 L}{2G}\int_0^s \frac{\mathrm{d}s}{t} = \tfrac{1}{2}T\theta.$$

and

$$\theta = \frac{TL}{4A^2G}\int_0^s \frac{\mathrm{d}s}{t}. \tag{2.7}$$

* This quantity, τt, is sometimes referred to as the *shear flow*; the term derives from considering an analogous fluid flow of local velocity τ around a closed tube of local diameter t.

Equation (2.7) is the general expression for the twist of a thin tube; often the term $\dfrac{4A^2}{\int_0^s \dfrac{ds}{t}}$ is called the *torsion constant* and denoted by J, so that

$$\frac{T}{J} = \frac{G\theta}{l}.$$

In the general case plane cross-sections do not remain plane. The loss of plane sections is called *warping*, and if in a practical situation some restraint is placed on the ends of the tube to maintain them flat, or if the torque varies along the length of the tube, then longitudinal direct stresses will be induced. However, the calculation of these *warping stresses* is beyond the scope of this introductory text and the reader is referred to the list on p. 27 for further reading. This form of torsion in which the warping stresses are excluded is known as *St Venant Torsion*, so named because of the original development of the analysis by B. de Saint-Venant in 1855.

2.3 Maximum loads for thin tubes

The calculation of the maximum torque that a thin tube can carry requires quite complicated analysis. On the basis that the cross-section remains undistorted the shear stress can reach the yielding stress of the material τ_y. But if the tube is sufficiently thin compared to its radius, it is possible for the tube to buckle and the cross-section to become distorted before the yield stress is reached. In general there is little information available on the buckling behaviour of tubes with non-circular cross-sections. For circular tubes an experimental relationship is

$$\tau_b = k \frac{\pi^2 E}{24(1-\nu^2)} \left(\frac{t}{L}\right)^2,$$

where τ_b is the elastic buckling stress, E is Young's modulus and ν is Poisson's ratio. The factor k is related to the geometry of the tube by $k = \left[\dfrac{L(1-\nu^2)^{\frac{1}{4}}}{Rt}\right]^{\frac{1}{2}}$. Thus, we can describe the behaviour by saying that the circular tube will sustain the torque according to the relationships (2.2) and (2.3) until the torque reaches the buckling value $T_b = \tau_b 2\pi Rt$, when it will become very deformed as shown in Fig. 2.5.

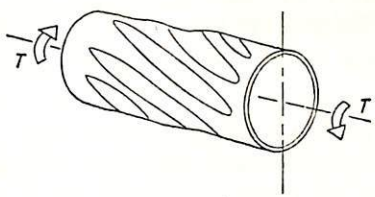


Fig. 2.5

Example

Compare the torque that can be carried by a circular tube of diameter 0.5 m and by a square tube having a width 0.5 m. Each has a wall thickness of 30 mm, is 2 m long and has a maximum allowable shearing stress of 35 N/mm². Also compare the stiffnesses of the tubes.

For the circular tube, from (2.1),

$$T = 2\pi R^2 t\tau = 2\pi \times (250)^2 \times 30 \times 35 \times 10^{-6} = 412 \text{ kN m}.$$

$$\text{Stiffness} \equiv \frac{T}{\theta} = \frac{G}{L} 2ARt = \frac{G}{L} \times 2\pi \times (250)^3 \times 30$$

$$= 2.95 \frac{G}{L}.$$

For the square tube,

$$T = 2At\tau = 2 \times (500)^2 \times 30 \times 35 = 525 \text{ kN m}.$$

Because the thickness is uniform, equation (2.7) is:

$$\frac{T}{\theta} = \frac{G}{L} \frac{4A^2 t}{\int_0^s ds} = \frac{G}{L} \cdot \frac{4 \times (500)^4 \times 30}{2000} = 3.75 \frac{G}{L}.$$

PROBLEMS

Problem 2.1 A thin circular tube, of mean diameter 0.3 m and 1 m long, is made from a material having a shear modulus of 80 kN/mm². Calculate the thickness of the tube if it is to carry a torque of 50 kN m with a shear stress 30 N/mm². Determine the rate of twist and check that the torque is not sufficient to cause the tube to buckle. $E = 200$ kN/mm², $v = 0.25$.

(Ans: 11.8 mm)

Problem 2.2 A 2 m length of tube is formed from steel sheet 5 mm thick having a maximum working shear stress of 40 N/mm². The cross-section of the tube is in the form of an equilateral triangle with side-length 0.15 m. What is the maximum allowable torque to which the tube can be subjected?

(Ans: 3.9 kNm)

Problem 2.3 A steel tube, 0.3 m mean diameter and 10 mm thick is placed inside an aluminium tube of 0.5 m mean diameter. The ends of the tubes are connected so that when the assembly is subjected to a torque of 240 kNm both tubes will have an equal rate of twist. Calculate the thickness of the aluminium tube that will ensure the shear stress developed in the steel tube is 70 N/mm².

$G_{aluminium} = 30$ kN/mm², $G_{steel} = 80$ kN/mm².

(Ans: 8.2 mm)

Problem 2.4 A tube, 2 m mean diameter and 4 mm thick, is to carry a maximum torque of 1.25 kN m. Calculate the maximum length of the tube. $E = 200$ kN/mm², $v = 0.25$.

(Ans: 3 m)

3

THICK-WALLED SHAFTS SUBJECT TO TORSION

This chapter is concerned with the manner in which thick-walled or solid structural members withstand torsional loading. These types of members are most common in mechanical engineering applications and include shafts for marine propellers, power transmission shafts in gear boxes, coil springs. Usually because of easy manufacture most of these shafts are circular but occasionally other cross-sectional shapes are used; for example, splined drive shafts.

3.1 Thick-walled circular shaft

Consider the shaft with circular cross-section shown in Fig. 3.1; we cannot consider this to be thin-walled and therefore we must expect the stresses to vary across the thickness although, from symmetry, it is apparent that they will not vary round the periphery. We make the following

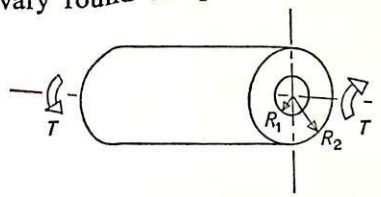


Fig. 3.1

assumption to simplify the analysis; since the shaft is free from any longitudinal restraint and subject only to a torque, we will assume that **the only stresses induced in the shaft are shear stresses and these will vary in such a way that a straight line across a section before loading will remain straight after loading.**

Taking an elementary tube of thickness dr, distance r from the centre, it will have shear stress τ applied to its cross-section and will deform similarly to the thin-walled tube analysed in § 2.1. Thus, as shown in Fig. 3.2, the elementary torque is

$$dT = 2\pi r^2 \tau dr, \tag{3.1}$$

and from the assumption of radii remaining straight after loading we find that the cross-section of the shaft will remain undeformed and merely

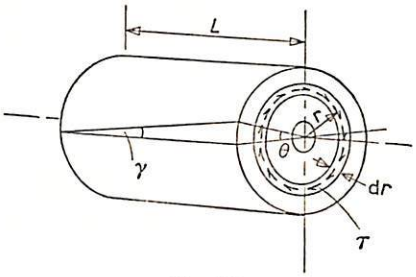


Fig. 3.2

rotate about the axis through an angle θ. Thus the twist of one end of any elementary tube relative to the other is proportional to the radius, that is:

$$r\theta = L\gamma,$$

and so the strain γ, and consequently the stress, is proportional to the distance from the axis. That is

$$\tau = \frac{G\theta r}{L}. \tag{3.2}$$

Substituting (3.2) into (3.1), the torque applied to the elementary tube is

$$\mathrm{d}T = 2\pi r^3 \frac{G\theta}{L}\,\mathrm{d}r$$

and

$$T = \frac{G\theta}{L}\int_{R_1}^{R_2} 2\pi r^3\,\mathrm{d}r. \tag{3.3}$$

The integral $\int_{R_1}^{R_2} 2\pi r^3\,\mathrm{d}r$ is the geometric quantity we earlier defined as the torsion constant J; in this particular case J is the Polar Second Moment of Area about the axis:

$$J = \frac{\pi}{2}[R_2^4 - R_1^4].$$

Thus from (3.1) and (3.2) we have

$$\frac{T}{J} = \frac{G\theta}{L} = \frac{\tau}{r}. \tag{3.4}$$

Notice that the maximum shear stress occurs at the outer surface, and when the shaft is solid, that is $R_1 = 0$, the torsion constant is $J = \frac{\pi}{2}R_2^4$ and the material at the centre is free from any shear stress.

Example

A hollow shaft has outer radius of 0.3 m, inner radius of 0.25 m, and is subject to a torque of 400 kN m. Determine the inaccuracy involved in using the theory for thin-walled tubes.

First we consider the maximum stress. From thick-walled tube theory we know that the stress will be greatest at the outer surface. That is,

$$\tau_{max} = \frac{TR}{J}; \quad J = \frac{\pi}{2}(0.3^4 - 0.25^4) = 0.006\,6\ \text{m}^4$$

$$\tau_{max} = \frac{400 \times 0.3 \times 10^{-3}}{0.006\,6} = 18.2\ \text{N/mm}^2.$$

The thin-walled theory assumes that the stress is constant through the thickness and is

$$\tau = \frac{T}{2At} = \frac{400 \times 10^{-3}}{2\pi \times 0.275^2 \times 0{\cdot}05} = 16.9\ \text{N/mm}^2,$$

that is it underestimates the true maximum stress by 8%.

The shaft stiffness derived using thick-walled theory is

$$\frac{T}{\theta} = \frac{GJ}{L} = 0.006\,6\frac{G}{L};$$

while from thin-walled theory we have

$$\frac{T}{\theta} = 2ARt\frac{G}{L} = 2\pi \times 0.275^3 \times 0.05\frac{G}{L}$$

$$= 0.006\,5\frac{G}{L}.$$

Thus for stiffness we have close agreement, and it may be shown generally that provided the ratio of thickness to mean radius is less than $\frac{1}{5}$, the thin-walled tube theory will predict the maximum stress with less than 10% error.

3.2 Combined stresses

In this book we deal mainly with developing the analysis whereby the shear stresses in elements subjected to torsion may be calculated. But it must not be forgotten that other stresses can be present in the structural component owing to the application of bending moments or direct forces. For example, the propeller shaft of a ship will be subject to the torque from the driving machinery as well as the thrust from the propeller. Another example of combined loading is a drive shaft supported between widely spaced bearings. Here the shaft is subjected to a bending moment due to its own weight in addition to the applied torque.

For the thick and thin-walled tubes considered in previous sections the results of the various loadings will be a system of plane stresses; these are represented in Fig. 3.3. From these stresses we must now determine the principal stresses σ_p in order to be in a position to make any decision regarding the failure of a shaft subject to combined loading. The principal stresses may be calculated using the formulation*

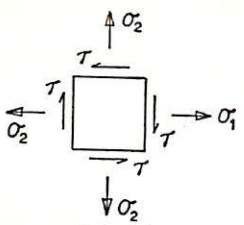


Fig. 3.3

$$\sigma_p = \tfrac{1}{2}(\sigma_1+\sigma_2) \pm \tfrac{1}{2}\sqrt{(\sigma_1-\sigma_2)^2+4\tau^2}.$$

Example

A solid circular propeller shaft is to transmit a torque of 900 kN m and a thrust of 7 MN. Calculate the radius required to limit the working direct stress to 80 N/mm^2.

The maximum shear stress due to the torque will occur at the outer surface of the shaft and is

$$\tau = \frac{900\times10^3\times R}{\frac{1}{2}\pi R^4} = \frac{0.572\times10^6}{R^3}\ \text{N/m}^2.$$

The compressive direct stress due to the thrust is

$$\sigma = -\frac{7\times10^6}{\pi R^2} = -\frac{2.23\times10^6}{R^2}\ \text{N/m}^2.$$

The maximum direct stress is the principal stress σ_p,

$$\sigma_p = \tfrac{1}{2}\sigma - \tfrac{1}{2}\sqrt{\sigma^2+4\tau^2},$$

hence

$$-80\times10^6 = -\frac{2.23\times10^6}{2R^2} - \tfrac{1}{2}\sqrt{\left(\frac{2.23\times10^6}{R^2}\right)^2+4\left(\frac{0.572\times10^6}{R^3}\right)^2}$$

or

$$80R^3 - 1.11R - \sqrt{1.243R^2+0.327} = 0.$$

This equation is most easily solved graphically to give the radius $R = 0.22$ m.

3.3 Non-circular shaft

The general analysis of solid shafts of arbitrary cross-section is complicated and involves the solution of partial differential equations. It is therefore outside the scope of this introductory text and all we can do here is to present some simple indications of the behaviour of typical cross-section shapes.

The first point of note is that except for circular cross-sections we must

* See T. R. Graves Smith, *Stress and Strain* (Chatto & Windus).

expect that plane sections will not remain plane. For example, consider the square shaft in Fig. 3.4 (a) twisted about its longitudinal axis. There can be no complementary stresses along the outside surfaces of the element A, and therefore no shear stress can be generated on that portion of the cross section. But on a portion such as B, shear stress can occur parallel to the boundary, hence the side *ac* can deform as in Fig. 3.4 (b); allowing these elements to become infinitesimally small, we have a smooth variation

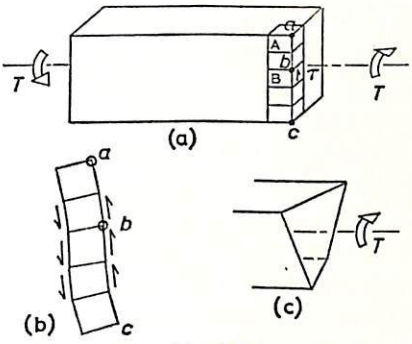


Fig. 3.4

of shear stress along the side *ac* and the cross-section will be seen to *warp* out of the plane as shown in Fig. 3.4 (c). If this longitudinal displacement is prevented by end constraints, or if the torque varies along the length of the shaft, longitudinal stress will be induced that must be included in the analysis. The circular shaft is free from this complication by virtue of its radial symmetry.

With present-day computers and methods of numerical analysis it is not too difficult to obtain a sufficiently accurate solution to the differential equations governing the variation of shear stress across a section and hence derive the corresponding torsion constant. Previously this was generally impossible and experimental methods were used. One of these is called the *soap-film analogy*, which still provides a good visual aid to the stress distribution in sections. It happens that the equations governing the distribution of shear stress in a section subject to torsion and those describing the shape of a soap bubble, or some other thin elastic membrane, having the same boundary shape as the shaft cross-section, are exactly analogous in form. Thus if we imagine, or construct, a soap-film on the boundary shape and subject it to small pressure then the slope of the surface is an analogous indication of the shear stress, in a direction normal to the direction in which the slope is measured, of the corresponding shaft section.

(a) *Circular section*

For example, suppose we have a circular shaft, the bubble with circular boundary will form a sphere, the slope of this will be radially symmetrical (as shown by circular contour lines in Fig. 3.5 (a)) and will become zero at the centre. This is precisely analogous to the shear stress derived in § 3.2 for this section.

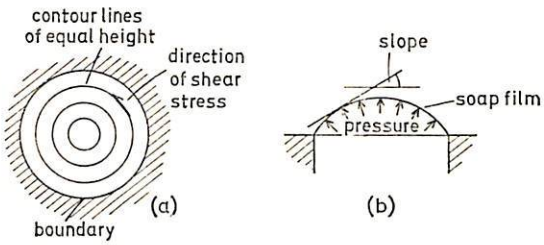


Fig. 3.5

(b) *Square section*

The contour lines indicate that the slope, and hence the shear, is greatest at the mid-point of the sides.

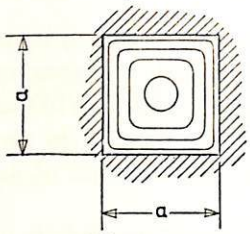


Fig. 3.6

Also the bubble is flat at the corners thus confirming our earlier discussion that the shear stress must indeed be zero at the corner.

The maximum value for shear stress is $\tau = 0.6\,\frac{T}{a^3}$ and the torsion constant (which is analogous to the volume under the bubble) is $J = 2.25\,a^4$.

(c) *Rectangular section*

Again, the closeness of the contour lines in Fig. 3.7 indicate that the greatest shear stress occurs at midpoint of the longer side and it has a value of $\tau = 0.25\,\frac{T}{a^3}$ for this section. The torsion constant is $7.32\,a^4$.

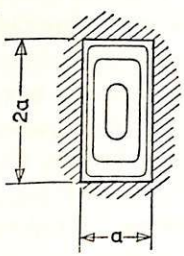


Fig. 3.7

It is valuable to consider what is the effect of introducing a notch or keyway into a shaft. This can be easily pictured by imagining the shape and contours of the equivalent soap film; we see in Fig. 3.8 that the contours are very closely bunched near the sharp corners at the bottom of the notch. This indicates very

high stress concentrations that have in the past led to failure through fatigue fracture. For this reason it is usual in practice to avoid sharp corners such as these in shafts subject to torsion.

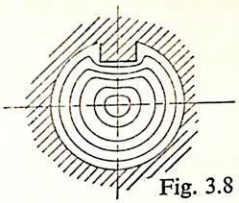


Fig. 3.8

3.4 Elastic-plastic material

In thick-walled tubes buckling is not a problem, and it is possible for such a component to carry a torque greater than that to cause the most highly stressed material to reach the yield stress. For a general cross-section the analysis of the behaviour of a shaft in which the material is partially plastic is very complicated. For simplicity, and also because it is a common practical geometry, we shall restrict attention to circular cross-sections. Also we shall assume that **the material is of the type that is linearly elastic up to a yield stress τ_y and thereafter yields extensively for no significant increase of stress.** This is shown in an idealised form in Fig. 3.9 (a). Consider the cross-section of a circular shaft subject to a torque T; we shall make the assumption that, **irrespective of whether**

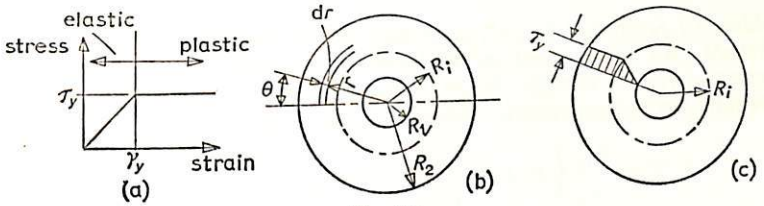


Fig. 3.9

the material is elastic or plastic, radii remain straight during loading. Thus, the shear strain at any radius r is $\gamma = \dfrac{r\theta}{L}$; provided the material is wholly elastic the analysis in § 2.1 will apply. But when the strain on the outer surface of the shaft exceeds the yield strain γ_y the stress cannot increase beyond the yield stress and the situation in Fig. 3.9 (c) occurs. The torque resultant of the stresses is

$$T = \int_{R_1}^{R_1} 2\pi\tau\, r^2\, \mathrm{d}r + \int_{R_1}^{R_2} 2\pi\tau_y\, r^2\, \mathrm{d}r.$$

But, in the elastic portion $\tau = \dfrac{G\theta r}{L}$

$$T = \int_{R_1}^{R_1} \frac{2\pi G\theta}{L}\, r^3\, \mathrm{d}r + \int_{R_1}^{R_2} 2\pi\tau_y\, r^2\, \mathrm{d}r. \tag{3.5}$$

The radius R_i of the elastic-plastic interface is

$$R_i = \frac{L}{G\theta}\tau_y,$$

so that

$$T = \frac{\pi}{2}\frac{\tau_y}{R_i}(R_i^4 - R_1^4) + \frac{2\pi}{3}\tau_y(R_2^3 - R_i^3). \tag{3.6}$$

The torque T_y at which the outer material will just yield occurs when $R_i = R_2$, and the maximum plastic torque T_p is when all the material is plastic, that is $R_i = R_1$.

From (3.6) the ratio of these is

$$\frac{T_p}{T_y} = \tfrac{4}{3}\frac{1-\left(\frac{R_1}{R_2}\right)^3}{1-\left(\frac{R_1}{R_2}\right)^3},$$

so that for a solid shaft, $R_1 = 0$, the maximum torque is 33% greater than that first to cause yield.

Problems

Problem 3.1 A composite shaft consisting of two concentric cylinders of different materials may be assumed to be strained by a torque in such a manner that diameters remain straight. Show that when there is no initial strain in the assembly, the twisting elastic strength is given by one of the expressions:

$$T = \frac{2\tau_1}{d_1}\left(J_1 + J_2\frac{G_2}{G_1}\right), \quad \text{or} \quad T_2 = \frac{2\tau_2}{d_2}\left(J_2 + J_1\frac{G_1}{G_2}\right),$$

where τ_1, τ_2 are the allowable shear stresses in the two materials; d_1, d_2 are the outer diameters of the individual cylinders, J_1, J_2 are the torsion constants and G_1, G_2 are the shear moduli.

Problem 3.2 A solid shaft must not twist more than 0.3 deg/m and the maximum shearing stress must not exceed 35 N/mm². If $G = 45$ kN/mm² what should the diameter be so that these two requirements are both satisfied? What would be the usable twisting strength of such a shaft?

(Ans: 296 mm; 0.28 kN m)

Problem 3.3 A composite shaft, consisting of 0.5 m length of aluminium alloy rod jointed to a 1 m length of titanium alloy rod, both of the same diameter 50 mm, is built into unyielding end supports. Where should an external torque T be applied in order that the allowable stresses of 35 N/mm² in the aluminium and 50 N/mm² in the titanium be just reached simultaneously? Also, what is the magnitude of the torque? For aluminium $G = 30$ kN/mm² for titanium $G = 45$ kN/mm.²

(Ans: 0.99 m from end of aluminium; 2.1 kN m)

Problem 3.4 A steel solid shaft 2 m long and 100 mm diameter. The material has a limiting working stress of 70 N/mm² and exhibits extensive yielding at a stress of 190 N/mm². What is the ratio of the maximum allowable working torque and the maximum torque the shaft could carry? $G = 80$ kN/mm².

(Ans: 0.28)

Problem 3.5 A drive shaft consists of a 2 m length of solid steel rod of diameter 150 mm joined to a 1 m length of square section rod in which the diagonal is 150 mm. Determine the stiffness of the shaft if it is loaded by a torque in such a way that it is free to warp. $G = 75$ kN/mm².

(Ans: 1714 kN m/rad.)

4

TORSION OF OPEN THIN-WALLED SECTIONS

An open section is one in which the centre line does not form a closed loop. Examples are the channel or I-sections used in structural engineering, and in this chapter we shall develop a theory, of sufficient accuracy for engineering purposes, to describe the behaviour of such components when they are subjected to torsion loading.

We start by considering the stress distribution in a thin strip of thickness t, breadth b and length L, shown in Fig. 4.1 (a). If we visualise the shape of a soap-film having the same boundary and being subject to a slight pressure, it will have zero slope in the x-direction (Fig. 4.1 (b)) for most

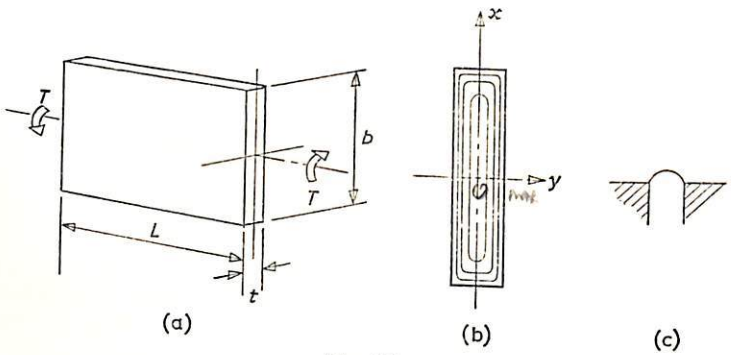


Fig. 4.1

of its length and will have zero slope at mid-thickness and increasing slope in the y-direction towards the boundaries. Hence we can deduce that the shear stress will be predominantly in the x-direction and will increase from zero at mid-thickness to a maximum at the centre of the long side.

Now we can make the assumption that **there will be no distortion in the shape of the cross-section during loading and that the section as a whole will rotate through an angle** θ. Considering an elementary tube of thickness dr, in Fig. 4.2 (a), which follows the shape of a constant shear stress contour, we approximate this by the idealised tube shown in Fig. 4.2 (b).

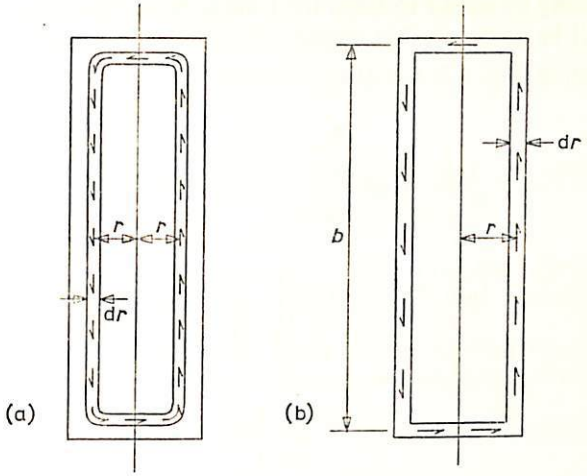


Fig. 4.2

Now from the theory developed in § 2.2, the torque dT on this tube is

$$dT = 2At\,\tau = 4br\,\tau\,dr. \tag{4.1}$$

Also the twist is

$$\theta = \frac{L\,dT}{8b^2r^2G}\,\frac{(b+2r)}{dr}. \tag{4.2}$$

If we neglect r in comparison to b in (4.2), we can eliminate dT/dr from (4.1) and (4.2) to obtain

$$\tau = 2r\,\frac{G\theta}{L}. \tag{4.3}$$

Hence

$$T = \int_0^{\frac{1}{2}t} 8br^2\,\frac{G\theta}{L}\,dr$$

$$T = \tfrac{1}{3}bt^3\,\frac{G\theta}{L}.$$

see p.11

We can now identify the torsion constant J as $J = \frac{1}{3}bt^3$. Equation (4.3) indicates that the shear stress varies linearly from the mid-thickness to the surface (this is in accordance with Fig. 4.1 (c) where for most of the length of the soap-film the shape would be semi-circular); the maximum stress is $\tau_{max} = t\,\frac{G\theta}{l}$.

This is an approximate theory that provides predictions that agree well with experimental results. We can extend the analysis to other open thin-walled shapes merely by noting that the torques taken by each element of

the section may be added to form the total torque. Since the rotation and material will be common this implies that torsion constants are additive. For example, in Fig. 4.3 $J = \frac{1}{3}(b_1 t_1^3 + 2b_2 t_2^3)$.

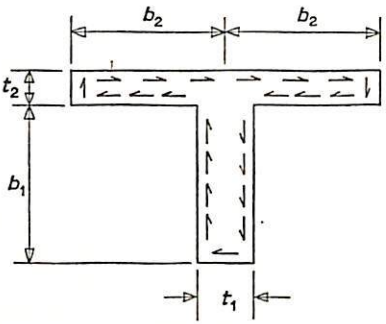


Fig. 4.3

Example

A thin-walled circular tube, 0.15 m mean radius and 1.5 m long is made by rolling sheet metal, 5 mm thick, to shape and then welding along the seam. Compare the stiffness of the tube before and after welding and, assuming the tubes were to work at the same stress of 35 N/mm² with no warping restraint, compare the torques they could carry. $G = 80$ kN/mm².

If the tube is closed,

$$T = 2A\tau t = 2\pi \times (150)^2 \times 35 \times 10^{-6} \times 5 = 24.6 \text{ kN m}.$$

$$\frac{T}{\theta} = \frac{4A^2Gt}{L\int_0^s ds} = \frac{4 \times \pi^2 \times (150)^4 \times 80 \times 5 \times 10^{-6}}{1\,500 \times 2\pi \times 150} = 5.7 \text{ MN m/rad}.$$

If the tube is open

$$T = \frac{\tau J}{t}; \quad J = \tfrac{1}{3}bt^3$$

where b is the length of the centre line. In this case $b = 2\pi R$

$$T = \frac{35 \times 2\pi \times 150 \times (5)^3 \times 10^{-6}}{3 \times 5} = 0.275 \text{ kN m}$$

$$\frac{T}{\theta} = \frac{GJ}{L} = \frac{80 \times 2\pi \times 150 \times (5)^3 \times 10^{-3}}{3 \times 1\,500} = 2.1 \text{ kN m/rad}.$$

TEXTS FOR FURTHER READING

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The theory of thick prismatic shafts is developed in the next section; emphasis is once more on the particularly simple, but nevertheless practically important, case of circular cross-section. By means of the soap-film analogy the discussion of the theory is extended to include the behaviour of non-circular cross-sections. The effects of material plasticity is also included in the analysis of the circular shafts.

The final section deals with open thin-walled structural elements. In this, as throughout the book, simplifying assumptions are made that enable the various theories to be developed without recourse to advanced mathematical analysis. In this way the mechanics of the problem are brought to the fore, and the student will have a better understanding of the physical behaviour of structural elements subjected to torsion.

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